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TWO MONOTONIC FUNCTIONS DEFINED BY TWO DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION

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Dedicated to my father, Mr. Shu-Gong Qi, on the occasion of his 80th birthday

Abstract. In the paper, by virtue of the convolution theorem for the Laplace transforms, with the help of monotonicity and logarithmic concavity of a function involving exponential function, and by means of analytic techniques, the author presents necessary and sufficient conditions for two functions defined by two derivatives of a function involving trigamma function to be completely monotonic or monotonic.

Keywords: complete monotonicity, monotonicity, necessary and sufficient condition, trigamma function, derivative, convolution theorem, Laplace transforms, exponential function, logarithmic concavity, inequality.

AMS Subject Classification: 26A48, 26A51, 26D07, 33B15, 44A10.

1. Introduction

In the literature [1, Section 6.4], the function $\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \,\mathrm{d}\, t$ for $\Re(z) > 0$ and its logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ are respectively called Euler's gamma function and digamma function. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for $k \in \mathbb{N}$ are known as polygamma functions.

Recall from Chapter XIII in [3], Chapter 1 in [11], and Chapter IV in [12] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, then we call $f(x)$ a completely monotonic function on I. Theorem 12b in [12, p. 161] characterized that a function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty \mathrm{e}^{-xt} \,\mathrm{d}\, \sigma(t), \quad x \in (0, \infty), \tag{1}$$

where $\sigma(s)$ is non-decreasing and the integral in (1) converges for $x \in (0, \infty)$. The integral representation (1) means that a function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of a non-decreasing measure $\sigma(s)$ on $(0, \infty)$.

Let $\Phi(x) = x\psi'(x) - 1$ on $(0, \infty)$. It is easy to see that

$$\Phi^{(k)}(x) = k\psi^{(k)}(x) + x\psi^{(k+1)}(x), \quad k \in \mathbb{N}.$$

In [7, Section 4] and [10, Theorem 4], it was proved that,

(1) the function $\mathfrak{H}_\alpha(x) = \Phi'(x) + \alpha\Phi^2(x)$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \ge 2$;
(2) the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \le 1$;

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(3) the double inequality $-2 < \frac{\Phi'(x)}{\Phi^2(x)} < -1$ is valid on $(0,\infty)$ and sharp in the sense that the lower bound -2 and the upper bound -1 cannot be replaced by any bigger number and any smaller number respectively.

Let

$$G(x) = x\Phi(x) - \frac{1}{2} = x\big[x\psi'(x) - 1\big] - \frac{1}{2} = x^2\left[\psi'(x) - \frac{1}{x}\right] - \frac{1}{2}$$

on $(0,\infty)$. It is easy to obtain that

$$G'(x) = \Phi(x) + x\Phi'(x) = x^2\psi''(x) + 2x\psi'(x) - 1$$

and

$$G^{(\ell)}(x) = \ell\Phi^{(\ell-1)}(x) + x\Phi^{(\ell)}(x) = x^2\psi^{(\ell+1)}(x) + 2\ell x\psi^{(\ell)}(x) + \ell(\ell-1)\psi^{(\ell-1)}(x)$$

for $\ell \ge 2$. For $k \in \{0\} \cup \mathbb{N}$ and $\theta_k, \tau_k \in \mathbb{R}$, let

$$\mathcal{G}_{k,\theta_k}(x) = G^{(2k+1)}(x) + \theta_k\big[G^{(k)}(x)\big]^2 \tag{2}$$

and

$$\mathfrak{G}_{k,\tau_k}(x) = \frac{G^{(2k+1)}(x)}{\big[(-1)^k G^{(k)}(x)\big]^{\tau_k}} \tag{3}$$

on $(0,\infty)$. In this paper, we will find necessary and sufficient conditions on θ_k and τ_k such that

(1) the functions $\pm\mathcal{G}_{k,\theta_k}(x)$ are completely monotonic on $(0,\infty)$;
(2) the function $\mathfrak{G}_{k,\tau_k}(x)$ is monotonic on $(0,\infty)$.

2. Lemmas

The following lemmas are necessary in this paper.

Lemma 2.1. *Let*

$$w(t) = \begin{cases} \dfrac{e^t[(t-2)e^t + t + 2]}{(e^t - 1)^3}, & t \ne 0; \\ \dfrac{1}{6}, & t = 0. \end{cases}$$

Then the following conclusions are valid:

(1) *the function $w(t)$ is decreasing from $(0,\infty)$ onto $\big(0,\frac{1}{6}\big)$;*
(2) *the function $w(t)$ is logarithmically concave on $(-\infty,\infty)$;*
(3) *the function $\frac{w(2t)}{w^2(t)}$ is even on $(-\infty,\infty)$, decreasing from $(0,\infty)$ onto $(0,6)$, increasing from $(-\infty,0)$ onto $(0,6)$;*
(4) *for any fixed $t > 0$, the function $w(st)w((1-s)t)$ is increasing in $s \in \big(0,\frac{1}{2}\big)$.*

Proof. It is not difficult to see that

$$w'(t) = -\frac{e^t\big[e^{2t}(t-3) + 4e^t t + t + 3\big]}{(e^t - 1)^4} = -\frac{e^t}{(e^t - 1)^4}\left(\frac{t^5}{30} + \sum_{k=6}^{\infty}\big[(k-6)2^{k-1} + 4k\big]\frac{t^k}{k!}\right) < 0$$

on $(0,\infty)$. Hence, the function $w(t)$ is decreasing on $(0,\infty)$.

Straightforward computation yields

$$[\ln w(t)]'' = -\frac{e^{4t} - 4\big(t^2 - 3t + 4\big)e^{3t} - \big(4t^2 - 30\big)e^{2t} - 4\big(t^2 + 3t + 4\big)e^t + 1}{(e^t - 1)^2[(t-2)e^t + t + 2]^2}$$
$$= -\frac{\displaystyle\sum_{k=8}^{\infty}\begin{pmatrix} 29 \times 2^{2k-5} - 4\big(k^2 - 10k + 36\big)3^{k-2} \\ +2^{2k-7} - 4\big(k^2 + 2k + 4\big) \\ +\big(11 \times 2^{k-7} - k^2 + k + 30\big)2^k \end{pmatrix}\frac{t^k}{k!}}{(e^t - 1)^2[(t-2)e^t + t + 2]^2}$$

$$= -\frac{\frac{t^8}{180} + \frac{t^9}{90} + \frac{109t^{10}}{9450} + \frac{13t^{11}}{1575} + \frac{2071t^{12}}{453600} + \frac{157t^{13}}{75600} + \frac{10573t^{14}}{13097700} + \cdots}{(\mathrm{e}^t - 1)^2[(t-2)\mathrm{e}^t + t + 2]^2}.$$

Since $29 \times 2^{2k-5} - 4\bigl(k^2 - 10k + 36\bigr)3^{k-2} > 0$ for $k \ge 7$, $2^{2k-7} - 4\bigl(k^2 + 2k + 4\bigr) > 0$ for $k \ge 8$, and $11 \times 2^{k-7} - k^2 + k + 30 > 0$ for $k \ge 9$, we obtain that $[\ln w(t)]'' < 0$ on $(0, \infty)$. This means that the function $w(t)$ is logarithmically concave on $(0, \infty)$.

Let $\varphi(t) = e^{4t} - 4\bigl(t^2 - 3t + 4\bigr)e^{3t} - \bigl(4t^2 - 30\bigr)e^{2t} - 4\bigl(t^2 + 3t + 4\bigr)e^t + 1$ on $[0, \infty)$. Direct differentiation differentiation yields

$$\begin{aligned}
\varphi'(t) &= 4\mathrm{e}^t\bigl[\mathrm{e}^{3t} - \mathrm{e}^{2t}\bigl(3t^2 - 7t + 9\bigr) - \mathrm{e}^t\bigl(2t^2 + 2t - 15\bigr) - t^2 - 5t - 7\bigr] \\
&\triangleq 4\mathrm{e}^t\varphi_1(t) \\
&\to 0, \quad t \to 0, \\
\varphi_1'(t) &= 3\mathrm{e}^{3t} - \mathrm{e}^{2t}\bigl(6t^2 - 8t + 11\bigr) - \mathrm{e}^t\bigl(2t^2 + 6t - 13\bigr) - 2t - 5, \\
\varphi_1''(t) &= 9\mathrm{e}^{3t} - 2\mathrm{e}^{2t}\bigl(6t^2 - 2t + 7\bigr) - \mathrm{e}^t\bigl(2t^2 + 10t - 7\bigr) - 2, \\
\varphi_1'''(t) &= \mathrm{e}^t\bigl[27\mathrm{e}^{2t} - 8\mathrm{e}^t\bigl(3t^2 + 2t + 3\bigr) - 2t^2 - 14t - 3\bigr] \\
&\triangleq \mathrm{e}^t\varphi_2(t) \\
&\to 0, \quad t \to 0, \\
\varphi_2'(t) &= 54\mathrm{e}^{2t} - 8\mathrm{e}^t\bigl(3t^2 + 8t + 5\bigr) - 4t - 14, \\
\varphi_2''(t) &= 108\mathrm{e}^{2t} - 8\mathrm{e}^t\bigl(3t^2 + 14t + 13\bigr) - 4, \\
\varphi_2'''(t) &= 8\mathrm{e}^t\bigl(27\mathrm{e}^t - 3t^2 - 20t - 27\bigr) \\
&\triangleq 8\mathrm{e}^t\varphi_3(t) \\
&\to 0, \quad t \to 0, \\
\varphi_3'(t) &= 27\mathrm{e}^t - 6t - 20 \\
&> 0
\end{aligned}$$

on $[0, \infty)$. As a result, we obtain that $\varphi_k^{(\ell)}(t) > 0$ for all $1 \le k \le 2$ and $1 \le \ell \le 3$ and $\varphi'(t) > 0$ on $(0, \infty)$. Further, since $\varphi(0) = 0$, we deduce $\varphi(t) > 0$ on $(0, \infty)$. This means $[\ln w(t)]'' < 0$ on $(0, \infty)$. The logarithmic concavity of $w(t)$ on $(0, \infty)$ is proved once again.

From $w(t) = w(-t)$, it follows that $w'(t) = -w'(-t)$. Accordingly, we acquire

$$[\ln w(-t)]' = -\frac{w'(-t)}{w(-t)} = \frac{w'(t)}{w(t)} = [\ln w(t)]'.$$

As a result, the function $w(t)$ is also logarithmically concave on $(-\infty, 0)$.

Direct computation gives

$$\begin{aligned}
\frac{w(2t)}{w^2(t)} &= \frac{2(\mathrm{e}^t - 1)^3\bigl[(t-1)\mathrm{e}^{2t} + t + 1\bigr]}{(\mathrm{e}^t + 1)^3[(t-2)\mathrm{e}^t + t + 2]^2} \\
&= \begin{cases} 6, & t \to 0; \\ 0, & t \to \pm\infty, \end{cases} \\
\left[\frac{w(2t)}{w^2(t)}\right]' &= -\frac{2(\mathrm{e}^t - 1)^2\begin{bmatrix} t\mathrm{e}^{5t} - \bigl(8t^2 - 17t + 12\bigr)\mathrm{e}^{4t} - 4\bigl(t^2 + 2t - 3\bigr)\mathrm{e}^{3t} \\ -4\bigl(t^2 - 2t - 3\bigr)\mathrm{e}^{2t} - \bigl(8t^2 + 17t + 12\bigr)\mathrm{e}^t - t \end{bmatrix}}{(\mathrm{e}^t + 1)^4[(t-2)\mathrm{e}^t + t + 2]^3} \\
&\triangleq -\frac{2(\mathrm{e}^t - 1)w_1(t)}{(\mathrm{e}^t + 1)^3[(t-2)\mathrm{e}^t + t + 2]^3}, \\
w_1'(t) &= \mathrm{e}^{5t}(5t + 1) - \mathrm{e}^{4t}\bigl(32t^2 - 52t + 31\bigr) - 4\mathrm{e}^{3t}\bigl(3t^2 + 8t - 7\bigr) \\
&\quad - 8\mathrm{e}^{2t}\bigl(t^2 - t - 4\bigr) - \mathrm{e}^t\bigl(8t^2 + 33t + 29\bigr) - 1
\end{aligned}$$

$$
\begin{aligned}
&\to 0, \quad t\to 0,\\
w_1''(t) &= \big[5\,\mathrm{e}^{4t}(5t+2) - 8\,\mathrm{e}^{3t}\big(16t^2-18t+9\big) - 4\,\mathrm{e}^{2t}\big(9t^2+30t-13\big)\\
&\quad - 8\,\mathrm{e}^{t}\big(2t^2-9\big) - 8t^2 - 49t - 62\big]\,\mathrm{e}^{t}\\
&\triangleq w_2(t)\,\mathrm{e}^{t}\\
&\to 0, \quad t\to 0,\\
w_2'(t) &= 5\,\mathrm{e}^{4t}(20t+13) - 8\,\mathrm{e}^{3t}\big(48t^2-22t+9\big) - 8\,\mathrm{e}^{2t}\big(9t^2+39t+2\big)\\
&\quad - 8\,\mathrm{e}^{t}\big(2t^2+4t-9\big) - 16t - 49\\
&\to 0, \quad t\to 0,\\
w_2''(t) &= 8\big[5\,\mathrm{e}^{4t}(10t+9) - \mathrm{e}^{3t}\big(144t^2+30t+5\big)\\
&\quad - \mathrm{e}^{2t}\big(18t^2+96t+43\big) - \mathrm{e}^{t}\big(2t^2+8t-5\big) - 2\big]\\
&\to 0, \quad t\to 0,\\
w_2'''(t) &= 8\big[10\,\mathrm{e}^{3t}(20t+23) - 9\,\mathrm{e}^{2t}\big(48t^2+42t+5\big)\\
&\quad - 2\,\mathrm{e}^{t}\big(18t^2+114t+91\big) - 2t^2 - 12t - 3\big]\,\mathrm{e}^{t}\\
&\triangleq 8w_3(t)\,\mathrm{e}^{t}\\
&\to 0, \quad t\to 0,\\
w_3'(t) &= 10\,\mathrm{e}^{3t}(60t+89) - 36\,\mathrm{e}^{2t}\big(24t^2+45t+13\big)\\
&\quad - 2\,\mathrm{e}^{t}\big(18t^2+150t+205\big) - 4t - 12\\
&\to 0, \quad t\to 0,\\
w_3''(t) &= 30\,\mathrm{e}^{3t}(60t+109) - 36\,\mathrm{e}^{2t}\big(48t^2+138t+71\big) - 2\,\mathrm{e}^{t}\big(18t^2+186t+355\big) - 4\\
&\to 0, \quad t\to 0,\\
w_3'''(t) &= 2\big[135\,\mathrm{e}^{2t}(20t+43) - 72\,\mathrm{e}^{t}\big(24t^2+93t+70\big) - 18t^2 - 222t - 541\big]\,\mathrm{e}^{t}\\
&\triangleq 2w_4(t)\,\mathrm{e}^{t}\\
&\to 448, \quad t\to 0,\\
w_4'(t) &= 6\big[45\,\mathrm{e}^{2t}(20t+53) - 12\,\mathrm{e}^{t}\big(24t^2+141t+163\big) - 6t - 37\big]\\
&\to 2352, \quad t\to 0,\\
w_4''(t) &= 36\big[15\,\mathrm{e}^{2t}(20t+63) - 2\,\mathrm{e}^{t}\big(24t^2+189t+304\big) - 1\big]\\
&\to 12096, \quad t\to 0,\\
w_4'''(t) &= 72\big[15\,\mathrm{e}^{t}(20t+73) - 24t^2 - 237t - 493\big]\,\mathrm{e}^{t}\\
&\triangleq 72w_5(t)\,\mathrm{e}^{t}\\
&\to 43344, \quad t\to 0,\\
w_5'(t) &= 3\big[5\,\mathrm{e}^{t}(20t+93) - 16t - 79\big]\\
&\to 1158, \quad t\to 0,\\
w_5''(t) &= 15\,\mathrm{e}^{t}(20t+113) - 48\\
&\to 1647, \quad t\to 0,\\
w_5'''(t) &= 15\,\mathrm{e}^{t}(20t+133)\\
&> 0
\end{aligned}
$$

on $(0,\infty)$. This means that $w_k^{(\ell)}(t)>0$ for $1\le\ell\le3$ and $2\le k\le5$ on $(0,\infty)$, that $w_1''(t)>0$ and $w_1'(t)>0$ on $(0,\infty)$, and that $w_1(t)>0$ on $(0,\infty)$. Hence, the first derivative $\bigl[\frac{w(2t)}{w^2(t)}\bigr]'$ is negative on $(0,\infty)$. As a result, the function $\frac{w(2t)}{w^2(t)}$ is decreasing on $(0,\infty)$.

From the evenness of $\frac{w(2t)}{w^2(t)}$ on $(-\infty,\infty)$, it follows that the function $\frac{w(2t)}{w^2(t)}$ is increasing on $(-\infty,0)$.

Direct differentiation results in

$$\begin{aligned}
\frac{\mathrm{d}[w(st)w((1-s)t)]}{\mathrm{d}\,s}&=tw'(st)w((1-s)t)-tw(st)w'((1-s)t)\\
&=tw(st)w((1-s)t)\biggl[\frac{w'(st)}{w(st)}-\frac{w'((1-s)t)}{w((1-s)t)}\biggr]\\
&=tw(st)w((1-s)t)\biggl[\frac{\mathrm{d}\ln w(s)}{\mathrm{d}\,s}\biggr|_{s=st}-\frac{\mathrm{d}\ln w(s)}{\mathrm{d}\,s}\biggr|_{s=(1-s)t}\biggr]\\
&>0
\end{aligned}$$

for $0<s<\frac12$, where we used the fact that $st<(1-s)t$ for $0<s<\frac12$ and the fact that $w(t)$ is logarithmically concave on $(-\infty,\infty)$. Accordingly, for any fixed $t>0$, the function $w(st)w((1-s)t)$ is increasing in $s\in\bigl(0,\frac12\bigr)$. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *For $k\in\{0\}\cup\mathbb{N}$, the function $(-1)^kG^{(k)}(x)$ is completely monotonic on $(0,\infty)$, with the limits*

$$\lim_{x\to0^+}\bigl[(-1)^kG^{(k)}(x)\bigr]=\begin{cases}\dfrac12, & k=0\\ 1, & k=1\\ (-1)^kk(k-1)\psi^{(k-1)}(1), & k\ge2\end{cases}\tag{4}$$

and

$$\lim_{x\to\infty}\bigl[(-1)^kx^{k+1}G^{(k)}(x)\bigr]=\frac{k!}{6}.\tag{5}$$

Proof. In the proof of [10, Theorem 4], the second author established that

$$G(x)=\int_0^\infty w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t.$$

This means

$$(-1)^kG^{(k)}(x)=\int_0^\infty w(t)t^k\,\mathrm{e}^{-xt}\,\mathrm{d}\,t,\tag{6}$$

which is completely monotonic on $(0,\infty)$.

For $\Re(z)>0$ and $k\in\mathbb{N}$, we have

$$\psi^{(k-1)}(z+1)=\psi^{(k-1)}(z)+(-1)^{k-1}\frac{(k-1)!}{z^k}.$$

See [1, p. 260, 6.4.6]. By this recurrence relation, we obtain

$$\begin{aligned}
G(x)&=x\biggl[x\biggl(\psi'(x+1)+\frac{1}{x^2}\biggr)-1\biggr]-\frac12\\
&\to\frac12,\quad x\to0^+,\\
G'(x)&=x^2\biggl[\psi''(x+1)-\frac{2}{x^3}\biggr]+2x\biggl[\psi'(x+1)+\frac{1}{x^2}\biggr]-1\\
&\to-1,\quad x\to0^+,
\end{aligned}$$

and, when $\ell \ge 2$,

$$
\begin{aligned}
G^{(\ell)}(x) &= x^2\left[\psi^{(\ell+1)}(x+1)-(-1)^{\ell+1}\frac{(\ell+1)!}{x^{\ell+2}}\right]+2\ell x\left[\psi^{(\ell)}(x+1)-(-1)^{\ell}\frac{\ell!}{x^{\ell+1}}\right]\\
&\quad+\ell(\ell-1)\left[\psi^{(\ell-1)}(x+1)-(-1)^{\ell-1}\frac{(\ell-1)!}{x^{\ell}}\right]\\
&= -(-1)^{\ell+1}\frac{(\ell+1)!}{x^{\ell}}-2\ell(-1)^{\ell}\frac{\ell!}{x^{\ell}}-\ell(\ell-1)(-1)^{\ell-1}\frac{(\ell-1)!}{x^{\ell}}\\
&\quad+x^2\psi^{(\ell+1)}(x+1)+2\ell x\psi^{(\ell)}(x+1)+\ell(\ell-1)\psi^{(\ell-1)}(x+1)\\
&= x^2\psi^{(\ell+1)}(x+1)+2\ell x\psi^{(\ell)}(x+1)+\ell(\ell-1)\psi^{(\ell-1)}(x+1)\\
&\to \ell(\ell-1)\psi^{(\ell-1)}(1), \quad x\to 0^+.
\end{aligned}
$$

Three limits in (4) are thus proved.

In [1, p. 260, 6.4.11], it was given that, for $|\arg z|<\pi$, as $z\to\infty$,

$$
\psi^{(n)}(z)\sim(-1)^{n-1}\left[\frac{(n-1)!}{z^n}+\frac{n!}{2z^{n+1}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+n-1)!}{(2k)!z^{2k+n}}\right], \tag{7}
$$

where B_{2k} for $k\in\mathbb{N}$ stands for the Bernoulli numbers which are generated by

$$
\frac{z}{\mathrm{e}^z-1}=1-\frac{z}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!}, \quad |z|<2\pi.
$$

By virtue of the asymptotic expansion (7), as $x\to\infty$, we have

$$
\begin{aligned}
xG(x) &\sim x\left(x\left[x\left(\frac{1}{x}+\frac{1}{2x^2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{x^{2k+1}}\right)-1\right]-\frac{1}{2}\right)\\
&=\sum_{k=1}^{\infty}\frac{B_{2k}}{x^{2k-2}}\\
&\to B_2,\\
x^2G'(x) &\sim x^2\left[2x\left(\frac{1}{x}+\frac{1}{2x^2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{x^{2k+1}}\right)-x^2\left(\frac{1}{x^2}+\frac{1}{x^3}+\sum_{k=1}^{\infty}\frac{(2k+1)B_{2k}}{x^{2k+2}}\right)-1\right]\\
&=\sum_{k=1}^{\infty}\frac{(1-2k)B_{2k}}{x^{2k-2}}\\
&\to -B_2,
\end{aligned}
$$

and, for $\ell\ge 2$,

$$
\begin{aligned}
x^{\ell+1}G^{(\ell)}(x) &= x^{\ell+1}\left(x^2(-1)^{\ell}\left[\frac{\ell!}{x^{\ell+1}}+\frac{(\ell+1)!}{2x^{\ell+2}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+\ell)!}{(2k)!x^{2k+\ell+1}}\right]\right.\\
&\quad+2\ell x(-1)^{\ell-1}\left[\frac{(\ell-1)!}{x^{\ell}}+\frac{\ell!}{2x^{\ell+1}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+\ell-1)!}{(2k)!x^{2k+\ell}}\right]\\
&\quad\left.+\ell(\ell-1)(-1)^{\ell-2}\left[\frac{(\ell-2)!}{x^{\ell-1}}+\frac{(\ell-1)!}{2x^{\ell}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+\ell-2)!}{(2k)!x^{2k+\ell-1}}\right]\right)\\
&=(-1)^{\ell}\sum_{k=1}^{\infty}\frac{(2k+\ell-2)!}{(2k-2)!}\frac{B_{2k}}{x^{2k-2}}\\
&\to(-1)^{\ell}\ell!B_2.
\end{aligned}
$$

Applying $B_2 = \frac{1}{6}$ to the above three limits and unifying lead to the limit (5). The proof of Lemma 2.2 is complete. $\square$

Lemma 2.3 (Convolution theorem for the Laplace transforms [12, pp. 91–92])**.** *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \ge 0$ such that $|f_k(t)| \le M_k \mathrm{e}^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u) \,\mathrm{d}\, u\right] \mathrm{e}^{-st} \,\mathrm{d}\, t = \int_0^\infty f_1(u) \,\mathrm{e}^{-su} \,\mathrm{d}\, u \int_0^\infty f_2(v) \,\mathrm{e}^{-sv} \,\mathrm{d}\, v.$$

Lemma 2.4 ([4, Theorem 6.1])**.** *If $f(x)$ is differentiable and logarithmically concave on $(-\infty, \infty)$, then the product $f(x) f(x_0 - x)$ for any fixed number $x_0 \in \mathbb{R}$ is increasing in $x \in \left(-\infty, \frac{x_0}{2}\right)$ and decreasing in $x \in \left(\frac{x_0}{2}, \infty\right)$.*

Lemma 2.5. *For $k, m \in \mathbb{N}$, the function*

$$V_{k,m}(x) = \frac{(1-x)^{k+m} + (1+x)^{k+m}}{(1-x)^k + (1+x)^k}$$

is increasing in $x \in [0, 1]$, with $V_{k,m}(0) = 1$ and $V_{k,m}(1) = 2^m$.

Proof. Let

$$U_{k,m}(x) = V_{k,m}\left(\frac{1}{x+1}\right).$$

Direct differentiation gives

$$\begin{aligned}
U'_{k,m}(x) &= -\frac{\begin{pmatrix} [(2k+m)x + 2k]\big[(x+2)^m - x^m\big] x^k (x+2)^k \\ -2m x^{k+m} (x+2)^k \\ +m x (x+2)\big[(x+2)^{2k+m-1} - x^{2k+m-1}\big] \end{pmatrix}}{x(x+2)(x+1)^{m+1}[x^k + (x+2)^k]^2} \\
&= -\frac{\begin{pmatrix} [(2k+m)x + 2k]\big[(x+2)^m - x^m\big] x^k (x+2)^k \\ -2m x^{k+m} (x+2)^k \\ +2m x (x+2) \sum_{\ell=0}^{2k+m-2} (x+2)^\ell x^{2k+m-2-\ell} \end{pmatrix}}{x(x+2)(x+1)^{m+1}[x^k + (x+2)^k]^2} \\
&= -\frac{\begin{pmatrix} [(2k+m)x + 2k]\big[(x+2)^m - x^m\big] x^k (x+2)^k \\ +2m \sum_{\substack{0 \le \ell \le 2k+m-2 \\ \ell \ne k-1}} (x+2)^{\ell+1} x^{2k+m-(\ell+1)} \end{pmatrix}}{x(x+2)(x+1)^{m+1}[x^k + (x+2)^k]^2} \\
&< 0
\end{aligned}$$

on $(0, \infty)$. Hence, the function $U_{k,m}(x)$ is decreasing on $[0, \infty)$.

Since $U_{k,m}(x)$ is decreasing on $[0, \infty)$, the function $V_{k,m}(x)$ is increasing on $[0, 1]$. The proof of Lemma 2.5 is complete. $\square$

3. Necessary and sufficient conditions for complete monotonicity

In this section, we find necessary and sufficient conditions on the scalar θ_k such that the functions $\pm \mathcal{G}_{k,\theta_k}(x)$ defined in (2) are completely monotonic on $(0, \infty)$.

Theorem 3.1. *For $k \in \{0\} \cup \mathbb{N}$ and $\theta_k \in \mathbb{R}$,*

1. *the function $\mathcal{G}_{k,\theta_k}(x)$ is completely monotonic on $(0, \infty)$ if and only if $\theta_k \ge \frac{3(2k+2)!}{k!(k+1)!}$;*
2. *the function $-\mathcal{G}_{k,\theta_k}(x)$ is completely monotonic on $(0, \infty)$ if and only if $\theta_k \le 0$.*

Proof. If the function $\mathcal{G}_{k,\theta_k}(x)$ is completely monotonic, then its first derivative is

$$\mathcal{G}'_{k,\theta_k}(x) = G^{(2k+2)}(x) + 2\theta_k G^{(k)}(x) G^{(k+1)}(x) \le 0,$$

that is,

$$\theta_k \ge -\frac{1}{2}\frac{G^{(2k+2)}(x)}{G^{(k)}(x)G^{(k+1)}(x)} = \frac{1}{2}\frac{(-1)^{2k+2}x^{2k+3}G^{(2k+2)}(x)}{[(-1)^k x^{k+1}G^{(k)}(x)][(-1)^{k+1}x^{k+2}G^{(k+1)}(x)]}$$
$$\to \frac{1}{2}\frac{\frac{(2k+2)!}{6}}{\frac{k!}{6}\frac{(k+1)!}{6}} = \frac{3(2k+2)!}{k!(k+1)!}$$

as $x \to \infty$, where we used the limit (5) in Lemma 2.2.

If the function $-\mathcal{G}_{k,\theta_k}(x)$ is completely monotonic, then its first derivative is

$$\mathcal{G}'_{k,\theta_k}(x) = G^{(2k+2)}(x) + 2\theta_k G^{(k)}(x)G^{(k+1)}(x) \ge 0,$$

that is,

$$\begin{aligned}
\theta_k &\le -\frac{1}{2}\frac{G^{(2k+2)}(x)}{G^{(k)}(x)G^{(k+1)}(x)} \\
&\to -\frac{1}{2}\times\begin{cases} \dfrac{2\psi'(1)}{\frac{1}{2}(-1)}, & k=0 \\ \dfrac{12\psi'''(1)}{(-1)2\psi'(1)}, & k=1 \\ \dfrac{(2k+2)(2k+1)\psi^{(2k+1)}(1)}{k(k-1)\psi^{(k-1)}(1)(k+1)k\psi^{(k)}(1)}, & k\ge 2 \end{cases} \\
&= \begin{cases} 2\psi'(1), & k=0 \\ \dfrac{3\psi'''(1)}{\psi'(1)}, & k=1 \\ -\dfrac{2k+1}{k^2(k-1)}\dfrac{\psi^{(2k+1)}(1)}{\psi^{(k-1)}(1)\psi^{(k)}(1)}, & k\ge 2 \end{cases}
\end{aligned}$$

as $x \to 0^+$, where we used the limits in (4).

By virtue of the integral representation (6) and Lemma 2.3, we have

$$\begin{aligned}
\mathcal{G}_{k,\theta_k}(x) &= \theta_k\left[\int_0^\infty w(t)t^k \,\mathrm{e}^{-xt}\,\mathrm{d}\,t\right]^2 - \int_0^\infty t^{2k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t \\
&= \int_0^\infty\left[\theta_k\int_0^t u^k(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u - t^{2k+1}w(t)\right]\mathrm{e}^{-xt}\,\mathrm{d}\,t.
\end{aligned} \tag{8}$$

From the logarithmic concavity of $w(t)$ in Lemma 2.1 and from Lemma 2.4, it follows that

$$\begin{aligned}
&\theta_k\int_0^t u^k(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u - t^{2k+1}w(t) \\
&\le \theta_k\int_0^t u^k(t-u)^k w\left(\frac{t}{2}\right)w\left(t-\frac{t}{2}\right)\mathrm{d}\,u - t^{2k+1}w(t) \\
&= \theta_k w^2\left(\frac{t}{2}\right)\int_0^t u^k(t-u)^k\,\mathrm{d}\,u - t^{2k+1}w(t) \\
&= \theta_k w^2\left(\frac{t}{2}\right)\frac{(k!)^2}{(2k+1)!}t^{2k+1} - t^{2k+1}w(t) \\
&= \left[\theta_k\frac{(k!)^2}{(2k+1)!} - \frac{w(t)}{w^2\left(\frac{t}{2}\right)}\right]t^{2k+1}w^2\left(\frac{t}{2}\right)
\end{aligned} \tag{9}$$

and

$$\theta_k\int_0^t u^k(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u - t^{2k+1}w(t) \ge \theta_k\int_0^t u^k(t-u)^k w(0)w(t)\,\mathrm{d}\,u - t^{2k+1}w(t)$$
$$= \theta_k\frac{(k!)^2}{(2k+1)!}t^{2k+1}w(0)w(t) - t^{2k+1}w(t) = \left[\frac{\theta_k}{6}\frac{(k!)^2}{(2k+1)!}-1\right]t^{2k+1}w(t), \quad (10)$$

where we used the computation

$$\int_0^t u^k(t-u)^k\,\mathrm{d}\,u = t^{2k+1}\int_0^1 s^k(1-s)^k\,\mathrm{d}\,s = B(k+1,k+1)t^{2k+1} = \frac{(k!)^2}{(2k+1)!}t^{2k+1}.$$

Applying the third conclusion in Lemma 2.1 to (9), when $\theta_k \le 0$, then

$$\theta_k\int_0^t u^k(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u - t^{2k+1}w(t) \le 0$$

which means that the function $-\mathcal{G}_{k,\theta_k}(x)$ is completely monotonic on $(0,\infty)$. This sufficient condition $\theta_k \le 0$ is trivial.

From (10), we can deduce that, when $\theta_k \ge \frac{6(2k+1)!}{(k!)^2} = \frac{3(2k+2)!}{k!(k+1)!}$,

$$\theta_k\int_0^t u^k(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u - t^{2k+1}w(t) \ge 0$$

which means that the function $\mathcal{G}_{k,\theta_k}(x)$ is completely monotonic on $(0,\infty)$.

The integral representation (8) can be alternatively rearranged as

$$\mathcal{G}_{k,\theta_k}(x) = \int_0^\infty\left[\theta_k\frac{\int_0^t u^k(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u}{t^{2k+1}w(t)}-1\right]t^{2k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t$$
$$= \int_0^\infty\left[\theta_k\frac{\int_0^1 s^k(1-s)^k w(st)w((1-s)t)\,\mathrm{d}\,s}{w(t)}-1\right]t^{2k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t.$$

Making use of the fact that $\lim_{t\to\infty}\frac{\mathrm{e}^t w(t)}{t} = 1$, we can deduce that

$$\lim_{t\to\infty}\frac{w(st)w((1-s)t)}{w(t)} = \lim_{t\to\infty}\left[\frac{\mathrm{e}^{st}w(st)}{st}\frac{\mathrm{e}^{(1-s)t}w((1-s)t)}{(1-s)t}\frac{t}{\mathrm{e}^t w(t)}s(1-s)t\right] = \infty$$

for $s\in(0,1)$. This means that, if $\theta_k > 0$, for $s\in(0,1)$, it is impossible to require the inequality

$$\theta_k\frac{\int_0^1 s^k(1-s)^k w(st)w((1-s)t)\,\mathrm{d}\,s}{w(t)} - 1 \le 0$$

to be valid in $t\in(0,\infty)$. Hence, by Lemma 1, if $\theta_k > 0$, the function $-\mathcal{G}_{k,\theta_k}(x)$ is surely not completely monotonic on $(0,\infty)$. In conclusion, the condition $\theta_k \le 0$ is necessary and sufficient for the function $-\mathcal{G}_{k,\theta_k}(x)$ to be completely monotonic on $(0,\infty)$. The proof of Theorem 3.1 is complete. □

4. Necessary and sufficient conditions for monotonicity

In this section, we find necessary and sufficient conditions on the scalar τ_k such that the function $\mathfrak{G}_{k,\tau_k}(x)$ defined in (3) is monotonic on $(0,\infty)$.

Theorem 4.1. *For $k\in\{0\}\cup\mathbb{N}$ and $\tau_k\in\mathbb{R}$,*

1. *the function $\mathfrak{G}_{k,\tau_k}(x)$ is decreasing on $(0,\infty)$ if and only if $\tau_k \ge 2$;*
2. *the function $\mathfrak{G}_{k,\tau_k}(x)$ is increasing on $(0,\infty)$ if $\tau_k \le 1$;*

(3) *the function* $\mathfrak{G}_{k,\tau_k}(x)$ *is increasing on* $(0,\infty)$ *only if*

$$\tau_k \le \begin{cases} \psi'(1), & k=0; \\ -\dfrac{\psi'''(1)}{\psi'(1)\psi''(1)}, & k=1; \\ \dfrac{k-1}{k}\dfrac{\psi^{(k-1)}(1)\psi^{(2k+1)}(1)}{\psi^{(k)}(1)\psi^{(2k)}(1)}, & k\ge 2; \end{cases}$$

(4) *the limits*

$$\lim_{x\to 0^+}\mathfrak{G}_{k,\tau_k}(x) = \begin{cases} -2^{\tau_0}, & k=0 \\ 6\psi''(1), & k=1 \\ \dfrac{2(2k+1)}{(k-1)^{\tau_k}k^{\tau_k-1}}\dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k\ge 2 \end{cases} \tag{11}$$

and

$$\lim_{x\to\infty}\mathfrak{G}_{k,\tau_k}(x) = \begin{cases} -\infty, & \tau_k>2 \\ -\dfrac{3(2k+2)!}{k!(k+1)!}, & \tau_k=2 \\ 0, & \tau_k<2 \end{cases} \tag{12}$$

are valid;

(5) *the double inequality*

$$-\frac{3(2k+2)!}{k!(k+1)!} < \mathfrak{G}_{k,2}(x) < \begin{cases} -4, & k=0 \\ 6\psi''(1), & k=1 \\ \dfrac{2(2k+1)}{(k-1)^2k}\dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k\ge 2 \end{cases} \tag{13}$$

is valid on $(0,\infty)$ *and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.*

Proof. If the function $\mathfrak{G}_{k,\tau_k}(x)$ is decreasing on $(0,\infty)$, then its first derivative

$$\mathfrak{G}_{k,\tau_k}'(x) = \frac{G^{(2k+2)}(x)\left[(-1)^kG^{(k)}(x)\right] - \tau_kG^{(2k+1)}(x)(-1)^kG^{(k+1)}(x)}{\left[(-1)^kG^{(k)}(x)\right]^{\tau_k+1}} \le 0$$

which is equivalent to

$$\begin{aligned} \tau_k &\ge \frac{G^{(k)}(x)G^{(2k+2)}(x)}{G^{(k+1)}(x)G^{(2k+1)}(x)} \\ &= \frac{\left[(-1)^kx^{k+1}G^{(k)}(x)\right]\left[(-1)^{2k+2}x^{2k+3}G^{(2k+2)}(x)\right.}{[(-1)^{k+1}x^{k+2}G^{(k+1)}(x)][(-1)^{2k+1}x^{2k+2}G^{(2k+1)}(x)]} \\ &\to \frac{k!(2k+2)!}{(k+1)!(2k+1)!}, \quad x\to\infty \\ &= 2, \end{aligned}$$

where we used the limit (5).

If the function $\mathfrak{G}_{k,\tau_k}(x)$ is increasing on $(0,\infty)$, then its first derivative $\mathfrak{G}'_{k,\tau_k}(x)\ge 0$ which is equivalent to

$$\tau_k \le \frac{G^{(k)}(x)G^{(2k+2)}(x)}{G^{(k+1)}(x)G^{(2k+1)}(x)} \to \begin{cases} \psi'(1), & k=0\\ -\dfrac{\psi'''(1)}{\psi'(1)\psi''(1)}, & k=1\\ \dfrac{k-1}{k}\dfrac{\psi^{(k-1)}(1)\psi^{(2k+1)}(1)}{\psi^{(k)}(1)\psi^{(2k)}(1)}, & k\ge 2 \end{cases}$$

as $x\to 0^+$, where we used the limit (4).

Making use of the integral representation (6) gives

$$\mathfrak{G}_{k,\tau_k}(x) = -\frac{\int_0^\infty w(t)t^{2k+1}\,\mathrm{e}^{-xt}\,\mathrm{d}\,t}{\left[\int_0^\infty w(t)t^k\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\right]^{\tau_k}}.$$

Since

$$\mathfrak{G}'_{k,\tau_k}(x) = \frac{\begin{pmatrix} \int_0^\infty t^{2k+2}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\int_0^\infty t^k w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\\ -\tau_k\int_0^\infty t^{2k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\int_0^\infty t^{k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\end{pmatrix}}{\left[\int_0^\infty t^k w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\right]^{\tau_k+1}},$$

to prove that the function $\mathfrak{G}_{k,\tau_k}(x)$ is decreasing on $(0,\infty)$, it is sufficient to show

$$\tau_k\int_0^\infty t^{2k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\int_0^\infty t^{k+1}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t \ge \int_0^\infty t^{2k+2}w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t\int_0^\infty t^k w(t)\,\mathrm{e}^{-xt}\,\mathrm{d}\,t. \tag{14}$$

By Lemma 2.3, the inequality (14) can be rearranged as

$$\begin{aligned}\tau_k\int_0^\infty&\left[\int_0^t u^{2k+1}(t-u)^{k+1}w(u)w(t-u)\,\mathrm{d}\,u\right]\mathrm{e}^{-xt}\,\mathrm{d}\,t\\ &\ge\int_0^\infty\left[\int_0^t u^{2k+2}(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u\right]\mathrm{e}^{-xt}\,\mathrm{d}\,t. \end{aligned}\tag{15}$$

Let

$$P_k(t)=\int_0^t u^{2k+1}(t-u)^{k+1}w(u)w(t-u)\,\mathrm{d}\,u$$

and

$$Q_k(t)=\int_0^t u^{2k+2}(t-u)^k w(u)w(t-u)\,\mathrm{d}\,u.$$

Then the inequality (15) can be rewritten as

$$\int_0^\infty Q_k(t)\left[\tau_k\frac{P_k(t)}{Q_k(t)}-1\right]\mathrm{e}^{-xt}\,\mathrm{d}\,t\ge 0. \tag{16}$$

Changing the variable $u=\frac{(1+v)t}{2}$ results in

$$\begin{aligned}\frac{P_k(t)}{Q_k(t)} &= \frac{\int_0^1\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k+1}w\left(\frac{1+v}{2}t\right)w\left(\frac{1-v}{2}t\right)\mathrm{d}\,v}{\int_0^1[(1-v)^{k+2}+(1+v)^{k+2}](1-v^2)^k w\left(\frac{1+v}{2}t\right)w\left(\frac{1-v}{2}t\right)\mathrm{d}\,v}\\ &\to\frac{\int_0^1\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k+1}\mathrm{d}\,v}{\int_0^1[(1-v)^{k+2}+(1+v)^{k+2}](1-v^2)^k\,\mathrm{d}\,v},\quad t\to 0^+\\ &=\frac{2^{3k+3}B(2k+2,k+2)}{2^{3k+3}B(2k+3,k+1)}\\ &=\frac12,\end{aligned}\tag{17}$$

where we used the first conclusion in Lemma 2.1 and used the formula

$$\int_0^1 \left[(1+x)^{\mu-1}(1-x)^{\nu-1}+(1+x)^{\nu-1}(1-x)^{\mu-1}\right] \mathrm{d}\, x = 2^{\mu+\nu-1}B(\mu,\nu) = 2^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \tag{18}$$

for $\Re(\mu), \Re(\nu) > 0$ in [2, p. 321, 3.214].

Let

$$\begin{aligned} S_k(t) &= \int_0^1 \left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k+1} w\left(\frac{1+v}{2}t\right) w\left(\frac{1-v}{2}t\right) \mathrm{d}\, v \\ &\quad - \frac{1}{2}\int_0^1 \left[(1-v)^{k+2}+(1+v)^{k+2}\right]\left(1-v^2\right)^{k} w\left(\frac{1+v}{2}t\right) w\left(\frac{1-v}{2}t\right) \mathrm{d}\, v \\ &= \int_0^1 T_k(v)\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k} w\left(\frac{1+v}{2}t\right) w\left(\frac{1-v}{2}t\right) \mathrm{d}\, v, \end{aligned}$$

where

$$T_k(v) = 1-v^2-\frac{1}{2}\frac{(1-v)^{k+2}+(1+v)^{k+2}}{(1-v)^k+(1+v)^k}$$

with $T_k(0)=\frac{1}{2}$ and $T_k(1)=-2$. By Lemma 2.5 for $m=2$, we see that the function $T_k(v)$ is decreasing on $[0,1]$ and has only one zero $v_0 \in (0,1)$. As a result, by virtue of the fourth conclusion in Lemma 2.1, we acquire

$$\begin{aligned} S_k(t) &= \int_0^{v_0} + \int_{v_0}^1 T_k(v)\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k} w\left(\frac{1+v}{2}t\right) w\left(\frac{1-v}{2}t\right) \mathrm{d}\, v \\ &> w\left(\frac{1+v_0}{2}t\right) w\left(\frac{1-v_0}{2}t\right) \int_0^{v_0} T_k(v)\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k} \mathrm{d}\, v \\ &\quad + w\left(\frac{1+v_0}{2}t\right) w\left(\frac{1-v_0}{2}t\right) \int_{v_0}^{1} T_k(v)\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k} \mathrm{d}\, v \\ &= w\left(\frac{1+v_0}{2}t\right) w\left(\frac{1-v_0}{2}t\right) \int_0^{1} T_k(v)\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k} \mathrm{d}\, v \\ &= 0, \end{aligned}$$

where we used the formula (18) to obtain

$$\begin{aligned} &\int_0^{1} T_k(v)\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k} \mathrm{d}\, v \\ &= \int_0^1 \left[(1+v)^{k+1}(1-v)^{2k+1}+(1+v)^{2k+1}(1-v)^{k+1}\right] \mathrm{d}\, v \\ &\quad - \frac{1}{2}\int_0^1 \left[(1+v)^{k}(1-v)^{2k+2}+(1+v)^{2k+2}(1-v)^{k}\right] \mathrm{d}\, v \\ &= 2^{3k+3}B(k+2,2k+2) - 2^{3k+2}B(k+1,2k+3) \\ &= 0. \end{aligned}$$

Consequently, considering the limit in (17), we conclude an inequality $\frac{P_k(t)}{Q_k(t)} > \frac{1}{2}$ for $t>0$, which is sharp in the sense that the lower bound $\frac{1}{2}$ cannot be replaced by any bigger scalar. This sharp inequality shows that the inequality (16) is valid for all $\tau_k \geq 2$. Accordingly, the condition $\tau_k \geq 2$ is sufficient for $G_{\tau_k}(x)$ to be decreasing on $(0,\infty)$.

It is easy to see that

$$\begin{aligned} &\left[(1-v)^k+(1+v)^k\right]\left(1-v^2\right)^{k+1} - \left[(1-v)^{k+2}+(1+v)^{k+2}\right]\left(1-v^2\right)^{k} \\ &\qquad = -2v\left(1-v^2\right)^k\left[(1+v)^k-(1-v)^k+v\left((1-v)^k+(1+v)^k\right)\right] < 0 \end{aligned}$$

for $v \in (0,1)$. Combining this negativity with the positivity of $w(t)$ on $(0,\infty)$, we deduce an inequality $0 < \frac{P_k(t)}{Q_k(t)} < 1$ on $(0,\infty)$. This means that, when $\tau_k \le 1$, the function $G_{\tau_k}(x)$ is increasing on $(0,\infty)$.

Utilizing the limits (4) and 5 in Lemma 2.2, we have

$$\lim_{x\to0^+} \mathfrak{G}_{k,\tau_k}(x) = \frac{\lim_{x\to0^+} G^{(2k+1)}(x)}{\left[(-1)^k \lim_{x\to0^+} G^{(k)}(x)\right]^{\tau_k}} = \begin{cases} -2^{\tau_0}, & k=0\\ 6\psi''(1), & k=1\\ \dfrac{2(2k+1)}{(k-1)^{\tau_k}k^{\tau_k-1}}\dfrac{\psi^{(2k)}(1)}{\left|\psi^{(k-1)}(1)\right|}, & k\ge2 \end{cases}$$

and

$$\begin{aligned}\lim_{x\to\infty} \mathfrak{G}_{k,\tau_k}(x) &= (-1)^{2k+1} \lim_{x\to\infty} x^{(k+1)(\tau_k-2)} \lim_{x\to\infty} \frac{(-1)^{2k+1}x^{2k+2}G^{(2k+1)}(x)}{\left[(-1)^k x^{k+1} G^{(k)}(x)\right]^{\tau_k}}\\ &= \begin{cases} -\infty, & \tau_k>2;\\ -6^{\tau_k-1}\dfrac{(2k+1)!}{(k!)^{\tau_k}}, & \tau_k=2;\\ 0, & \tau_k<2. \end{cases}\end{aligned}$$

The inequalities in (13) and their sharpness follow from monotonicity of the function $\mathfrak{G}_{\mu_k}(x)$ and the limits in (11) and (12) for $\tau_k = 2$. The proof of Theorem 4.1 is complete. □

Corollary 4.1. *Let $k \in \{0\} \cup \mathbb{N}$, $\tau_k \in \mathbb{R}$, and*

$$\boldsymbol{G}_k(x) = (-1)^k \left[\tau_k G^{(2k+1)}(x) G^{(k+1)}(x) - G^{(2k+2)}(x) G^{(k)}(x)\right]$$

on $(0,\infty)$. Then

(1) *the function $\boldsymbol{G}_k(x)$ is completely monotonic on $(0,\infty)$ if and only if $\tau_k \ge 2$;*
(2) *the function $-\boldsymbol{G}_k(x)$ is completely monotonic on $(0,\infty)$ if $\tau_k \le 1$.*

Proof. This follows from the proof of Theorem 4.1. □

5. Conclusions

In mathematics and mathematical sciences, one regards necessary and sufficient condition as the best. In this paper, we have presented four necessary and sufficient conditions in Theorem 3.1, Theorem 4.1, and Corollary 4.1. These four necessary and sufficient conditions, two sufficient conditions in Theorem 4.1 and Corollary 4.1, and one necessary condition in Theorem 4.1 are main conclusions of this paper.

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